◎ <u>Stochastic Process</u>

・**Def：A stochastic process is a family of random variables X(t)**

X：state space

X(ω), ωåΩ can be discrete (countable) or continuous (uncountable)

t：time index, can be discrete or continuous

X：{X(t), t∈T} is called a stochastic process

・**Four types of stochastic process**

(1) discrete state, discrete time

－ ex：number of mails received on the nth day of the year

X(t)：**第** t **天收到的** mail **數**

(2) discrete state, continuous time

－ ex：number of WWW access during (0, t)

X(t)：(0, t) **時間內有多少** access request

(3) continuous state, discrete time

－ ex：period of time you played BBS on the nth day of the year

X(t)：**第** t **天用** BBS **的時間**

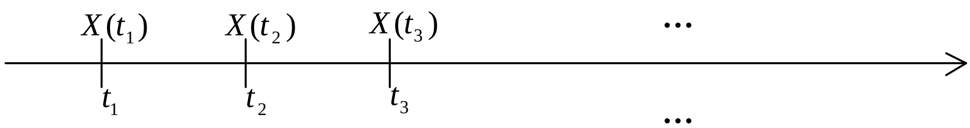
(4) continuous state, continuous time

－ ex：periods of time that a WWW server is busy during (0, t)

X(t)：(0, t) **時間內** server busy **的時間**

・**Relations of X(t) and X(t)**

X(t)：stochastic process



Define $F_x(\bar{x},\bar{t}) = P(X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_i) \le x_i)$, $\bar{x} = (x_1, x_2, \ldots)$, $\bar{t} = (t_1, t_2, \ldots)$

－if X is stationary： $F_x(\bar{x},\bar{t}) = F_x(\bar{x},\bar{t}+\bar{\boldsymbol{t}})$, $\bar{x} = (x_1, x_2, \ldots)$, $\bar{t} = (t_1 + \boldsymbol{t}, t_2 + \boldsymbol{t}, \ldots)$

－if X is independent： $f_x(\bar{x},\bar{t}) = f_x(x_1, t_1) * f_x(x_2, t_2) * \ldots$

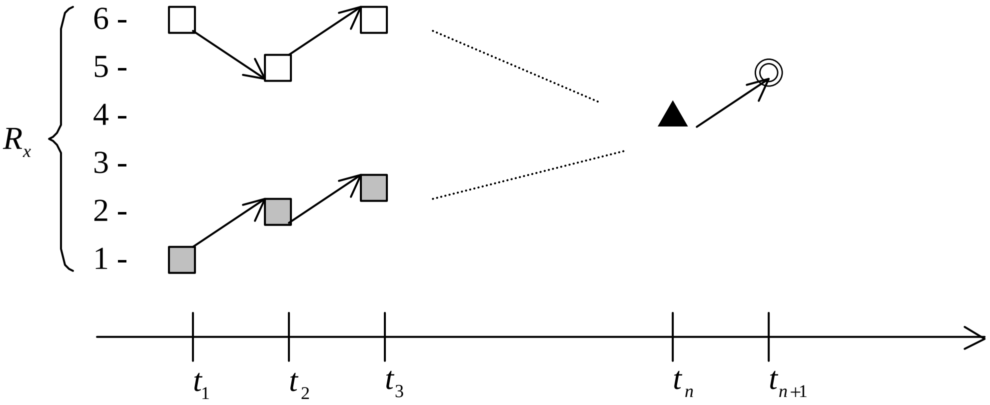
・**Markov Process**

future evolution of stochastic process depends only on current state

· **Def：** A discrete state Markov Process forms a Markov Chain (MC) if the probability of the next state depends only on current state

$$X = \{X(t_1), X(t_2), \ldots\} = \{x_1, x_2, \ldots\}$$

$$P(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1) = P(X_{n+1} = x_{n+1} | X_n = x_n)$$



· **Modeling using MC (Markov Chain)**

— State ： notation of a system state with transitions among states

ex ： number of jobs queued, number of available resources

· All relevant past history of system (for predicting future) must be contained in current state descriptor

— time ： discrete time v.s continuous time

· discrete time ： transitions between states occur only at discrete time

· continuous time ： transitions between states occur at any time

· **Discrete time MC**

— a discrete state, discrete time random process

— system has a possible set of countable states $\{x_1, x_2, \ldots\}$

— all past history summarized in current state

— transistions between states take place only at discrete time $(t_1, t_2, \ldots)$

— given MC is at state I, the probability the next state will be j is

$P[X_{n+1} = j | X_n = i] = P_{ij}$, $P = [P_{ij}]$ is called transition probability matrix

· **Example1：** weather

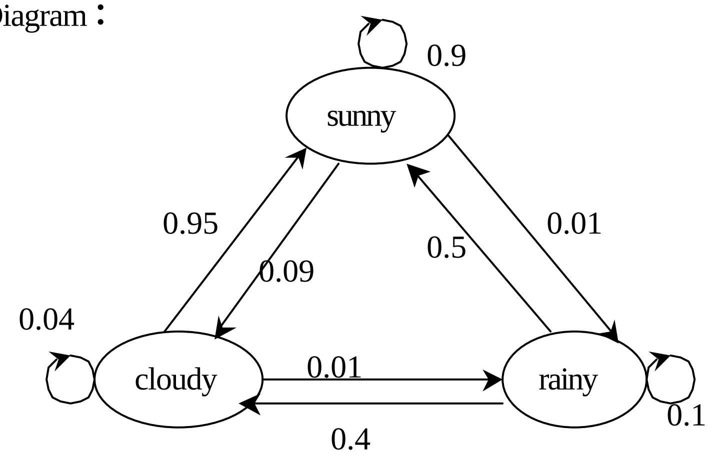
suppose tomorrow' s weather only depends on today' s weather

State = (sunny, cloudy, rainy)

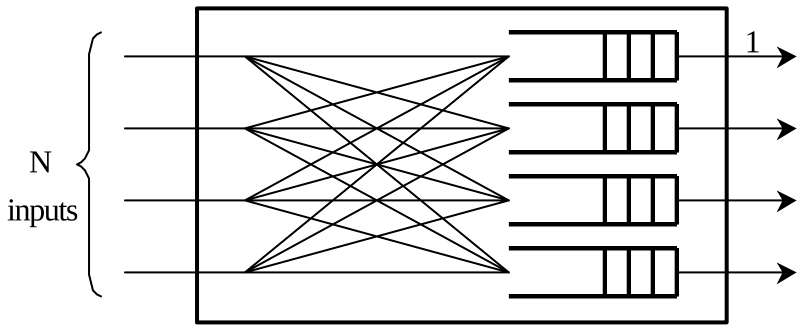
$$P = \begin{matrix} & S & C & R \\ S & 0.9 & 0.09 & 0.01 \\ C & 0.95 & 0.04 & 0.01 \\ R & 0.5 & 0.4 & 0.1 \end{matrix}$$

$$P = \begin{bmatrix} 0.9 & 0.09 & 0.01 \\ 0.95 & 0.04 & 0.01 \\ 0.5 & 0.4 & 0.1 \end{bmatrix}$$, if $\sum_{all j} P_{ij} = 1$, P is called a "stochastic" matrix

State Diagram：



・**Example2：** slotted concentrator with buffer (ATM)



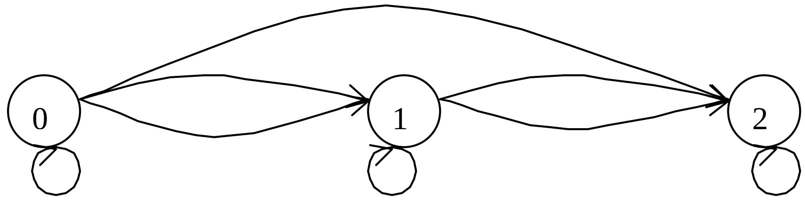
－ Model output port1, N inout ports, each port has an arrival with destination port1 with probability p during a slot time

－ All cells transmitted at an output port during each slot if there is any queued cells

Question：what is the expected delay？

－ time：discrete

－ state descriptor：queue length (number of cells in output buffer), $R_X = \{0,1,2,\ldots\}$



$P_{ij} =$ P(i + (# of attival) – (# of transition) = j | i cells in buffer)

ex：$P_{01} = \binom{N}{1} P(1-P)^{N-1}$

・ **Def：** m-setp transition probability

$$P_{ij}^{(m)} = p(X_{n+m} = j \mid X_n = i)$$
$$P_{ij}^{(1)} = p_{ij}$$
$$P_{ij}^{(m)} = \sum_k P_{ik}^{(m-1)} P_{kj} \text{ (forward chapman - kolmogov)}$$
$$= \sum_k P_{ik} P_{kj}^{(m-1)} \text{ (backward chapman - kolmogov))}$$
$$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{ij}^{(n)}$$

P = [P_{ij}] implies $P^{(m)}$ = m – step transition probability = [$P_{ij}^{(m)}$]

・ **Def：** Irreducible MC

A Markov Chain is irreducible if every state is reachable from any other state i.e $P_{ij}^{(m)}>0$ for some m,i,j $\in$ I

・ **Recurrence**

f_j = prob [ever returning to state j, given the system is in state j now]

$f_j^{(n)}$ = prob [first return to state j after n steps]

ex： $f_{jj}^1 = P_{jj}$

$$f_j = \sum_{i=1}^{\infty} f_j^i$$

－ if $f_j = 1$, state j is recurrent, state j will be visited infinitely often if system runs forever

－ if $f_j < 1$, state j is transient, each time we visite j, never return with probability 1- f_j

－ probability of return exactly n times = $f_j^{n-1}(1-f_j)$

－ E[# of returns to j] = $1/(1-f_j)$

－ A MC is recurrent iff all states are recurrent

$$M_j = \sum_{n=1}^{\infty} n * f_j^{(n)}$$

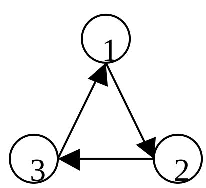
－ if $M_j < \infty$, state j is non - null recurrent

－ if $M_j = \infty$, state j is null – recurrent

・ **Periodicty**

If we can only return to state j after r,2r,3r,…transitions (r > 1), state j is periodic

ex：



· **Def**：Ergodic

A irreducible MC with all states are periodic, non – null recurrent is ergodic