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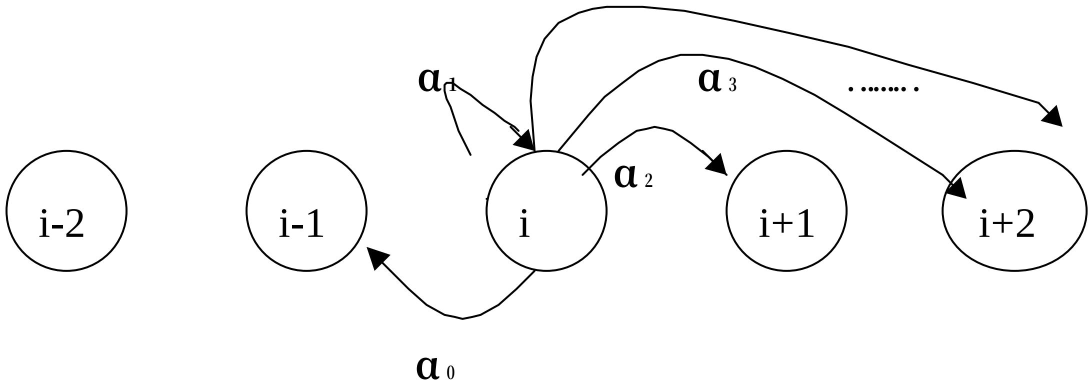
Form a DTMC embedded at departure instants of a M / G / 1 queue

state descriptor X：number of customers left behind



α_i＝prob(i arrivals during a service time)

$\alpha_k = \int_0^{\infty} b(x)$ P(k arrivals in time of x)dx

$$= \int_0^{\infty} b(x) \frac{(lx)^k e^{-lx}}{k!} dx$$

$$|P = \begin{matrix} a_0 & a_1 & a_2 & a_3 & \cdots & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & & & & \cdots \\ \vdots & \vdots & & & & \cdots \end{matrix}$$

$P = P \cdot |P$

$$\sum_{i=0}^{\infty} P_i = 1$$

Distribution of number of customers in system

利用 Z-transform：

$$Q(Z) = B^*(\lambda - Z)\frac{(1-r)(1-Z)}{B^*(l - lZ) - Z}$$

Distribution of delay

$$S^*(s) = B^*(s)\frac{s(1-r)}{s - l + lB^*(s)}$$

Distribution of waiting time

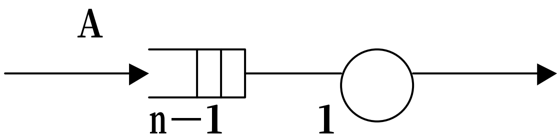
$$S^*(s) = W^*(s) \cdot B^*(s)$$

$$W^*(s) = \frac{1-r}{1 - r\hat{B}^*(s)}$$

$$\hat{B}^*(s) = \frac{1 - B^*(s)}{S\overline{X}}$$

E[N]＝expected number of customers in system

Method 1：Focus on an arriving customer A



λ：arrival rate

X：service time

P_i：prob (i customers in system)

W：waiting time seen by A

$$E[W] = \sum_{i=0}^{\infty} P_i \; E[W \mid i \text{ in system}]$$

$$= \sum_{i=0}^{\infty} P_i \; E[(i-1)\text{service time} + \text{residual service of customer in service}]$$

$$= \sum_{i=1}^{\infty} P_i \,(i-1)\,\overline{X} + \sum_{i=1}^{\infty} P_i \,\overline{R}$$

$$=\overline{Q}\,\overline{X}+(1-P_0)\overline{R}$$

$$E[W]=\lambda\,\overline{X}\,E[W]+\rho\,\overline{R}$$

$$E[W]=\frac{r\overline{R}}{1-r}$$

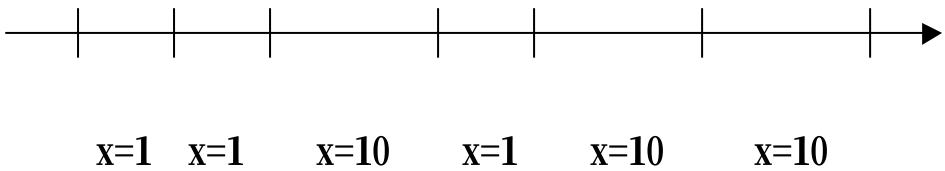
for M / M / 1

$$E[W]=\frac{r}{1-r}\times\frac{1}{m}$$

Finding $\overline{R}$:

suppose X is generally distributed with $P_x(i)=P(X=i)$

$$P(X=10)=\frac{1}{2},\quad P(X=1)=\frac{1}{2}$$



P(random observer sees a 10 period)

$\neq$ P(random observer sees a 1 period)

Let $N_i(T)$ = number of intervals of length i in [0,T]

P[random observer intercepts an i period]

$$=\frac{N_i(T)i}{\sum_{j=0}^{\infty}N_j(T)j}=\frac{\frac{N_i(T)}{N(T)}i}{\sum_{j=1}^{\infty}\frac{N_j(T)}{N(T)}j}=\frac{P_i}{\sum jP_j}i=\frac{iP_i}{\overline{X}}$$

In continuous case :

P(random observer intercepts x period)

$$=\frac{xf(x)}{\overline{x}}$$

Finding $\overline{R}$：

What is the expected remaining amount of time left for the job in service (seen by random (Poisson) arrival)?

$$E[R]=\int_0^\infty E[R \mid \text{service time is } x]\frac{xf(x)}{\overline{x}}dx$$

$$\overline{R}=\frac{E[X^2]}{2E[X]}=\int_0^\infty \frac{x^2 f(x)}{2\overline{x}}dx=\frac{E[x^2]}{2E[x]}$$

$$E[W]_{M/G/1}=\frac{r}{1-r}\overline{R}=\frac{rE[x^2]}{(1-r)2E[x]}$$

$$E[T]_{M/G/1}=E[W]_{M/G/1}+\overline{X}=\frac{rE[x^2]}{(1-r)2E[x]}+\frac{1}{m}$$

$$E[N]_{M/G/1}=E[T]\lambda=\frac{l}{m}+\frac{l\frac{l}{m}E[X^2]}{2(1-r)\frac{1}{m}}=\rho+\frac{l^2\overline{X}^2}{2(1-r)}$$

Def： $$C_b^2=\frac{E[X^2]-(E[X])^2}{(\overline{X})^2}$$

$$E[X^2]=\frac{(1+C_b{}^2)}{(E[X])^2}$$

$$E[T]_{M/G/1}=\frac{1}{m}+\frac{l}{2(1-r)m^2}(1+C_b{}^2)$$

$C_b=0$ deterministic

1 exponential

>1 hyper exponential

<1 Erlang