Continuous Time Makrov Chains

continuous time, discrete state

discrete time : transitions only at discrete time values

continuous time : transitions possible at any point in time

system has states $\{ 0, 1, 2, \ldots i, j, \ldots \}$

Def $\{ X(t), \forall t \geq 0 \}$ is a continuous time MC if $\forall s, t$, $\forall i, j$, $0 \leq v \leq s$

$$P[X(t+s) = i \mid X(s) = j \;\; AND \;\; X(v) = x(v) \;\; \forall v : 0 \leq v \leq s]$$

$$= P[X(t+s) = i \mid X(s) = j]$$

Def $P_{ij}(t) = P[X(t+s) = i \mid X(s) = j]$

indep. of s $\Rightarrow$ homogeneous **CTMC**

(1) time between transitions must be exponentially distributed

Forward Chapman-Kolmogorov Eq

$$P_{ij}(t+h) = \sum_{all \;\; k} P_{ik}(t) P_{kj}(h)$$

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \neq j} P_{ik}(t) P_{kj}(h) + P_{ij}(t) P_{jj}(h) - P_{ij}(t)$$

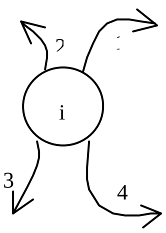
$$= \sum_{k \neq j} P_{ik}(t) P_{kj}(h) - P_{ij}(t)[1 - P_{jj}(h)]$$

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h \to 0} \sum_{k \neq j} \frac{P_{ik}(t) P_{kj}(h)}{h} - \frac{P_{ij}(t)[1 - P_{jj}(h)]}{h}$$

Def: transition rate from i to j

$$f(t) = q_{ij} e^{-q_{ij} t}$$

$$F(t) = 1 - e^{-q_{ij} t}$$



$$\sum_{j \neq i} q_{ij}$$

$$f_i(t) = \sum_{j \neq i} q_{ij} e^{-(\sum_{j \neq i} q_{ij}) t}$$

$$\lim_{h \to 0} \frac{P_{kj}(h)}{h} = \lim_{h \to 0} \frac{\int_0^h q_{kj} e^{-q_{kj} t} dt}{h} = \lim_{h \to 0} \frac{1 - e^{-q_{kj} h}}{h}$$

$$\lim_{h \to 0} \frac{1 - e^{-q_{kj} h}}{h} = \frac{1 - (1 - q_{kj} h)}{h} = q_{kj}$$

$$\lim_{h\to 0}\frac{1-P_{jj}(h)}{h}=\lim_{h\to 0}\frac{1-(1-\sum_{k\neq j}P_{jk}(h))}{h}=\lim_{h\to 0}\frac{\sum_{k\neq j}P_{jk}(h)}{h}=\sum_{k\neq j}q_{jk}$$

$$\therefore P'_{ij}(t)=\sum_{k\neq j}P_{ik}(t)q_{kj}-(\sum_{k\neq j}q_{jk})P_{ij}(t)$$

$$Q=[q_{ij}]\qquad q_{ij}=\begin{cases}=q_{ij} & i\neq j\\ =-\sum_{k\neq j}q_{jk} & if\ \ i=j\end{cases}$$

$$Q=\begin{array}{c} \\ 0\\ 1\\ 2\\ 3\\ \vdots\end{array}\begin{array}{c}\begin{array}{ccccc}0 & 1 & 2 & 3 & \cdots\end{array}\\ \begin{bmatrix}-\Sigma & q_{01} & q_{02} & q_{03} & \cdots\\ q_{10} & -\Sigma & q_{12} & q_{13} & \cdots\\ q_{20} & q_{21} & -\Sigma & q_{23} & \cdots\\ \vdots & & & & \\ \vdots & & & & \end{bmatrix}\end{array}$$

$$P'_{ij}(t)=\sum_{all\ \ k}P_{ik}(t)Q_{kj}$$

$$P_{ij}(t)=P(0)\cdot e^{-Qt}$$

$$P_{ij}(t)=\boldsymbol{p}_{ij}+\boldsymbol{s}_{ij}\cdot e^{-zt}$$

Steady State Distribution

Def $\boldsymbol{p}_j=\lim_{t\to\infty}P_{ij}(t)$

For a homogeneous, irreducible MC, $\boldsymbol{p}_j$ exist!

$$\lim_{t\to\infty}(\sum_{k\neq j}P_{ik}(t)q_{kj}-\sum_{k\neq j}q_{jk}P_{ij}(t))=0$$

$\boldsymbol{p}_k$ $\qquad$ $\boldsymbol{p}_j$

$$\sum_{k\neq j}\boldsymbol{p}_k q_{kj}-(\sum_{k\neq j}q_{jk})\boldsymbol{p}_j=0$$

$$\begin{cases}\boldsymbol{p}\cdot Q=0 & \boldsymbol{p}=\{\boldsymbol{p}_i\}\\ \sum_i\boldsymbol{p}_i=1\end{cases}$$

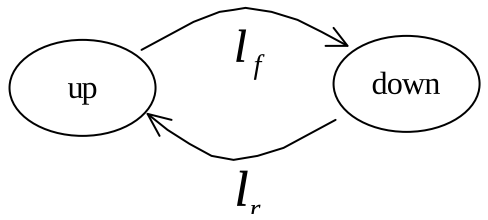
Q : infinitesimal generator or transition rate matrix

Example 1: On-Off source

(up/down server)

(a) time between failures is exponential distribution with mean $\frac{1}{l_f}$

(b) Repair time is exponential distribution with mean $\frac{1}{l_r}$

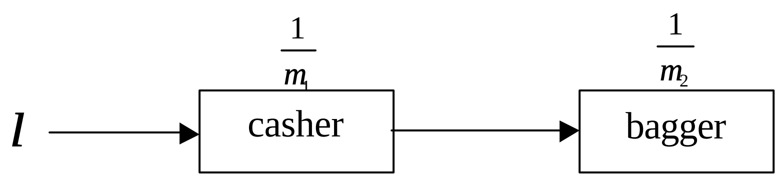


$$\begin{cases} p \cdot Q = 0 \\ \Sigma p = 1 \end{cases} \qquad Q = \begin{matrix} & up & down \\ up & -l_f & l_f \\ down & l_r & -l_r \end{matrix}$$

$$\begin{bmatrix} p_{up} & p_{down} \end{bmatrix} \begin{bmatrix} -l_f & l_f \\ l_r & -l_r \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad -l_f \cdot p_{up} + l_r \cdot p_{down} = 0$$

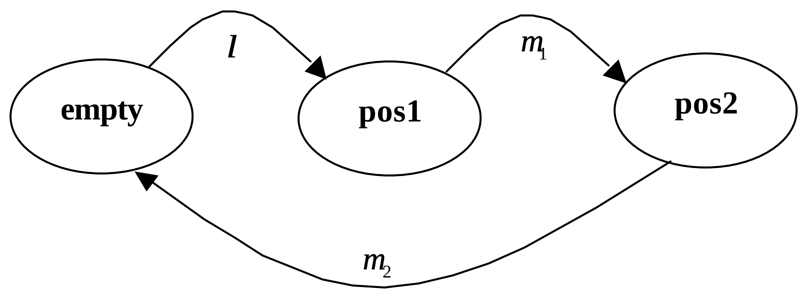
$$\begin{cases} p_{up} = \frac{l_r}{l_f} p_{down} \\ p_{up} + p_{down} = 1 \end{cases} \Rightarrow \begin{cases} p_{up} = \frac{l_r}{l_r + l_f} \\ p_{down} = \frac{l_f}{l_r + l_f} \end{cases}$$

Example 2: Supermarket checkout



State descriptor:

states: (**empty**, **pos1**, **pos2**)



$$
Q = \begin{array}{c} \\ e \\ c \\ b \end{array}
\begin{array}{c} \begin{array}{ccc} e & c & b \end{array} \\
\begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\mu_1 & \mu_1 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix} \end{array}
\qquad
\begin{cases} -\lambda \cdot \pi_{empty} + \mu_2 \cdot \pi_{pos2} = 0 \\ \lambda \cdot \pi_{empty} - \mu_1 \cdot \pi_{pos1} = 0 \\ \mu_1 \cdot \pi_{pos1} - \mu_2 \cdot \pi_{pos2} = 0 \\ \Sigma \pi_i = 0 \end{cases}
$$

$$
\pi_0 = \frac{\mu_1 \mu_2}{\mu_1 \mu_2 + \lambda(\mu_1 + \mu_2)}
$$

Example 3: Birth-Death Process

State of continuous time MC : {0, 1, 2, 3, ...}

$$
transitions \begin{cases} birth & k \to k+1 \\ death & k \to k-1 \end{cases}
$$

- time until next birth is exponential distribution with mean $\frac{1}{\lambda_k}$

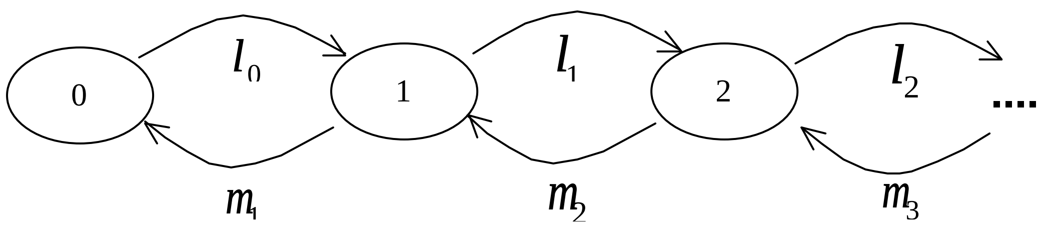
  $\Rightarrow$ *birth rate is* λ_k

- time until next death is exponential distribution with mean $\frac{1}{\mu_k}$

  $\Rightarrow$ *death rate is* μ_k

time stay at state k is exponential distribution with mean $\frac{1}{\lambda_k + \mu_k}$

$$
Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array}
\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{bmatrix}
-\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\
0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\
0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \cdots \\
0 & 0 & 0 & \mu_4 & -(\lambda_4 + \mu_4) & \lambda_{42} \\
\vdots & & & & & \cdots
\end{bmatrix} \end{array}
$$



$$
\begin{cases} \pi Q = 0 \\ \Sigma \pi_i = 1 \end{cases}
$$

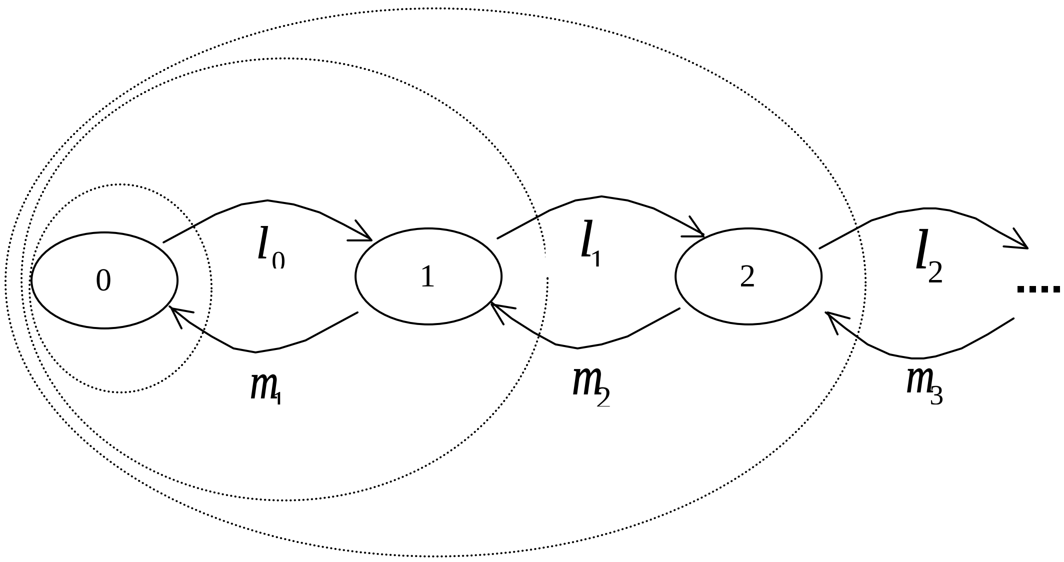
$$\begin{cases} -l_0 p_0 + m_1 p_1 = 0 \\ l_0 p_0 - (l_1 + m_1) p_1 + m_2 p_2 = 0 \\ \quad\vdots \\ l_{j-1} p_{j-1} - (l_j + m_j) p_j + m_{j+1} p_{j+1} = 0 \end{cases}$$

$$p_1 = \frac{l_0}{p_1} p_0 = r_0 p_0$$

$$\vdots$$

$$p_j = \left(\prod_{i=0}^{j-1} r_i\right) \cdot p_0$$

$$\boxed{r_i = \frac{l_i}{m_{i+1}}}$$



<u>flow in = flow out</u>

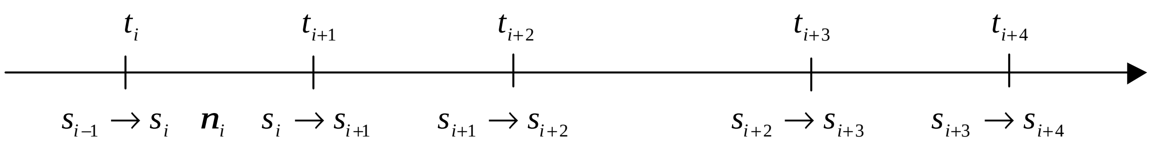
take state 2 as example

$$(l_2 + m_2) p_2 = l_1 p_1 + m_3 p_3$$

<u>Continuous Time MC V.S Discrete Time MC</u>

Y(t)

CTMC



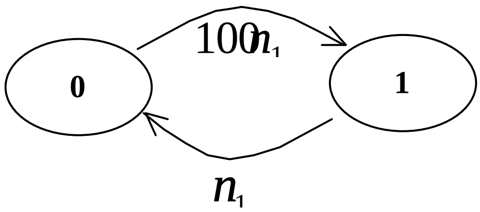
X = state of system immediately following a transition

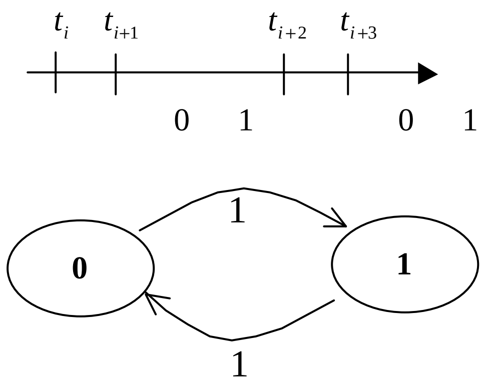
$X(t_i), \quad X(t_{i+1}), \quad X(t_{i+2}), \quad \cdots$

discrete time MC embedded at transition instants of **CTMC**

Example

CTMC **DTMC**





$p_0 \cdot 100 = p_1$

$p_0 = p_1 = 0.5$

$$\begin{cases} p_0 = \frac{1}{101} \\ p_1 = \frac{100}{101} \end{cases}$$

Let us consider every long period of time T, let

N : total number of state transitions within T

$\frac{1}{v_i}$: expected holding time in state i

n_i : number of transitions into i within T

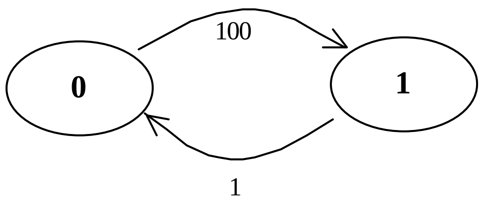
$$T \approx \Sigma n_i \frac{1}{v_i}$$

The average duration of time during T in state i is $\frac{n_i}{v_i}$

$$\Rightarrow p_i = \frac{\frac{n_i}{v_i}}{T} = \frac{\frac{n_i}{v_i}}{\lim_{n\to\infty} \Sigma_i \frac{n_i}{v_i}} = \frac{\frac{n_i}{N} \times \frac{1}{v_i}}{\lim_{n\to\infty} \Sigma_i \frac{n_i}{N} \cdot \frac{1}{v_i}} = \frac{\frac{p_i}{v_i}}{\Sigma_i \frac{p_i}{v_i}}$$

for above example

after modify, the result is



$p_0 = p_1 = 0.5 \quad v_0 = 100 \quad v_1 = 1$

$$p_0 = \frac{\frac{p_0}{v_0}}{\sum_i \frac{p_i}{v_i}} = \frac{0.5\times\frac{1}{100}}{0.5\times\frac{1}{100} + 1\times 0.5} = \frac{1}{101}$$

Uniformization

Purpose of uniformization

1. technique for computing $P_{ij}(t)$ in terms of transition matrix of embedded **DTMC**
2. analyze time dependent **CTMC** in terms of **DTMC** (useful computationally)

$$\frac{P_{ij}(t)}{dt} = \sum_{k\in E} P_{ik}(t) Q_{kj}$$

Example: let us consider a special **CTMC** in which all states have the same holding time , i.e. $v_i = v, \forall i$, let $X(t)$ is the state of **CTMC** at time t and $N(t)$ is the number of transitions occurred by time t which is a Poisson distribution with parameter v

$$P_{ij}(t) = P(X(t) = j \mid X(0) = i)$$

$$= \sum_{n=0}^{\infty} P(X(t) = j \mid X(0) = i \text{ and } N(t) = n) \cdot P(N(t) = n \mid X(0) = i)$$

$$= \sum_{n=0}^{\infty} P(X(t) = j \mid X(0) = i \text{ and } N(t) = n) \cdot (vt)^n \frac{e^{-vt}}{n!}$$

$$= \sum_{n=0}^{\infty} P_{ij}^{(n)} \cdot (vt)^n \frac{e^{-vt}}{n!}$$

Uniformization:

1. choose a v such that $v \ge v_i, \forall i$
2. create a uniformized **CTMC** with rates v and embedded **DTMC** with transition probability matrix P^*

$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^* e^{-vt} \frac{(vt)^n}{n!}$$

$$P_{ij}^* = \begin{cases} \dfrac{(1-\frac{v_i}{v})}{\frac{v_i}{v} P_{ij}} \end{cases}$$

$$P_{ij}^* \begin{cases} (1-\frac{v_i}{v}) & i = j \\ \frac{v_i}{v} P_{ij} & i \ne j \end{cases}$$

Supplement of Uniformization

1 DTMC vs CTMC

Let us look at a **CTMC** more carefully. Let S_i be the state immediately after transition i. Let t_i be the time of the occurrence of transition i. Note that the intervals between transitions, i.e., the holding time is state S_i, is exponentially distributed.

Define a stochastic process X such that

$$X = \{X(t_1), X(t_2), X(t_3), \cdots\}.$$

Stochastic process X is a **DTMC**! $\Rightarrow$ **DTMC** embedded at instants of **CTMC**.
Let

P_i: steady state probability of being in state i immediately following a transition(**DTMC**).

p_i: steady state probability of being in state i at any point in time(**CTMC**).

Question: Is $P_i = p_i$?

Answer: It is not always true because different state has different holding time.

Example: Consider a two state birth-death process with birth rate $100v$ and death rate v. First of all, we know

$$E[holding\ time\ in\ state\ 0] = \frac{1}{100} E[holding\ time\ in\ state\ 1]$$

Solve the steady state probability for this CTMS, we have $\pi_0 = \frac{1}{101}, \pi_1 = \frac{100}{101}$. However, solve the embedded **DTMC**, we have $P_0 = 0.5$, $P_1 = 0.5$. What is the relation between P_i and π_i?

Let us consider a very long period of time, T. Let

N : total number of state transitions within T.

$\frac{1}{v_i}$: expected holding time in state i.

n_i: number of transitions into i.

$$T \approx \sum_{i \in E} \frac{n_i}{v_i}$$

We also know that the average duration of time during T is in state i is $\frac{n_i}{v_i}$. Therefore, we have

$$\pi_i = \lim_{n \to \infty} \frac{\frac{n_i}{v_i}}{\sum_{i \in E} \frac{n_i}{v_i}} = \lim_{n \to \infty} \frac{\frac{n_i}{N v_i}}{\sum_{i \in E} \frac{n_i}{N v_i}} = \lim_{n \to \infty} \frac{\frac{P_i}{v_i}}{\sum_{i \in E} \frac{P_i}{v_i}}$$

2 Uniformization

The purpose of uniformization:

1. technique for computing $P_{ij}(t)$ in terms of transition matrix of embedded **DTMC**.
2. analyze time dependent **CTMC** in terms of **DTMC** (useful computationally).

$$\frac{P_{ij}(t)}{dt} = \sum_{k \in E} P_{ik}(t) Q_{kj}$$

Example: Let us consider a special **CTMC** in which all states have the same holding time, i.e. $v_i = v, \forall i$. Let $X(t)$ is the state of **CTMC** at time t and $N(t)$ is the number of transitions occurred by time t which is a Poisson distribution with parameter v.

$$P_{ij}(t) = P(X(t) = j \mid X(0) = i)$$

$$= \sum_{n=0}^{\infty} P(X(t) = j \mid X(0) = i \ and \ N(t) = n) \cdot P(N(t) = n \mid X(0) = i)$$

$$= \sum_{n=0}^{\infty} P(X(t) = j \mid X(0) = i \text{ and } N(t) = n) \cdot (vt)^n \frac{e^{-vt}}{n!}$$

$$= \sum_{n=0}^{\infty} P_{ij}^{(n)} \cdot (vt)^n \frac{e^{-vt}}{n!}$$

So we can solve $P_{ij}(t)$ easily via the **DTMC** transition matrix if all states have the same holding time. What if they do not have the same holding time? We need uniformization!

1. choose a v such that $v \geq v_i, \forall i$
2. create a uniformized **CTMC** with rates v and embedded **DTMC** with transition probability matrix P^*.

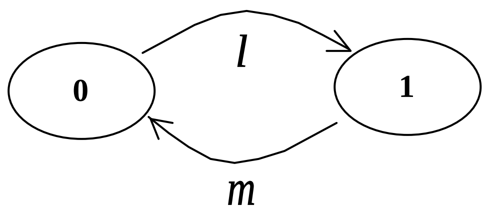
$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{*} e^{-vt} \frac{(vt)^n}{n!}$$

where

$$P_{ij}^{*} \begin{cases} (1 - \frac{v_i}{v}) & i = j \\ \frac{v_i}{v} P_{ij} & i \neq j \end{cases}$$

Example: Consider the two-state birth-death process again.

Answer: For the system, we have



We uniformize using $v = m + l$

$v_0 = l,, \quad P_{00} = 0, \quad P_{01} = 1$

$v_0 = l,, \quad P_{00} = 0, \quad P_{01} = 1$

$$P_{00}^{*} = P_{01}^{*} = m / (l + m)$$

$$P_{10}^{*} = P_{101}^{*} = l / (l + m)$$

It also turns out that the n-step transition as the matrix itself, i.e. $P_{ij}^{*(n)} = P_{ij}^{*}$

What is the probability of transmitting from state 0 to state 0?

$$P_{00}(t) = e^{-(m+l)t} + \sum_{n=1}^{\infty} P_{00}^{*} e^{-(m+l)t} \frac{[(m+l)t]^n}{n!} = e^{-(m+l)t} + [1 - e^{-(m+l)t}] \frac{m}{m+l} = \frac{m}{m+l} + \frac{l}{m+l} e^{-(m+l)t}$$

Similarly, the closed form expression for $P_{11}(t)$, the probability of remaining in state 1, is:

$$P_{11}(t) = e^{-(m+l)t} + \sum_{n=1}^{\infty} P_{11}^{*} e^{-(m+l)t} \frac{[(m+l)t]^n}{n!} = e^{-(m+l)t} + [1 - e^{-(m+l)t}] \frac{l}{m+l} = \frac{l}{m+l} + \frac{m}{m+l} e^{-(m+l)t}$$