Uniformization:

1. choose a v such that $v \geq v_i, \forall i$
2. create a uniformized **CTMC** with rates v and embedded **DTMC** with transition probability matrix P^*

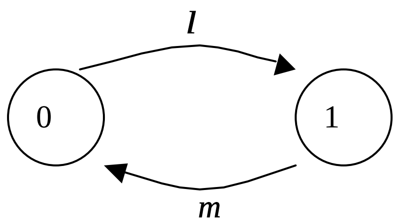
$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{*} e^{-vt} \frac{(vt)^n}{n!}$$

$$P_{ij}^{*} = \begin{cases} \dfrac{(1 - \frac{v_i}{v})}{\frac{v_i}{v} P_{ij}} \end{cases}$$

$$P_{ij}^{*} \begin{cases} (1 - \frac{v_i}{v}) & i = j \\ \frac{v_i}{v} P_{ij} & i \neq j \end{cases}$$

Example

two state birth-death process



$V_0 = l$ $\quad P_{00} = 0$

$V_1 = m$ $\quad P_{01} = 1$

$P_{10} = 1$

$P_{11} = 0$

Choice a $n = l + m$

$$P_{ij}^{*} = \begin{bmatrix} \frac{m}{l+m} & \frac{l}{l+m} \\ \frac{m}{l+m} & \frac{l}{l+m} \end{bmatrix}$$

$$P_{00}(t) = \sum_{n=0}^{\infty} P_{ij}^{*(n)} \cdot e^{-(l+m)t} \cdot \frac{[(l+m)t]^n}{n!}$$

$$= e^{-(l+m)t} + \sum_{n=1}^{\infty} \frac{[(l+m)t]^n}{n!} \cdot e^{-(l+m)t} \cdot P_{ij}^{*(n)}$$

$$= e^{-(l+m)t} + \left[e^{(l+m)t} - 1\right] \cdot e^{-(l+m)t} \cdot \frac{m}{l+m}$$

$$= \frac{m}{l+m} + \frac{l}{l+m} \cdot e^{-(l+m)t}$$

if $t \to \infty$

$$= \frac{m}{l+m}$$

可以利用 uniformization **求** transition **的** behaviors

Queuing Theory

- Study of systems which provide to customers may or may not allow customers to wait.
- To specify a queuing system
  - Characterize arrivals
    - Exponential (M)
    - Deterministic (D)
    - General (G)
  - Characterize service demands
    - M # of service
    - D # of service
    - G # of service
  - Queue
    - Queuing displines
    - Storage space (waiting space)

Kendoll' s Notation

A / B / n / K

- A: arrival
- B: service
- n: # of server
- Storage

(**沒寫表示無限大**)

eg:

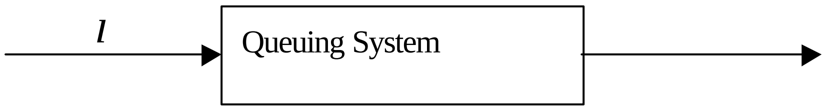
M/M/1 M/M/1/K M/G/1 M/G/1

Performance measure of internet

- avg waiting time
- avg number of customers (queue length)
- server utilization (fraction time server busy)
- through put (# of jobs /sec (unit of time))
- capacity (maximum through put)

For a stable system, through put =arrival rate

Two Fundamental Laws of Queuing Theory



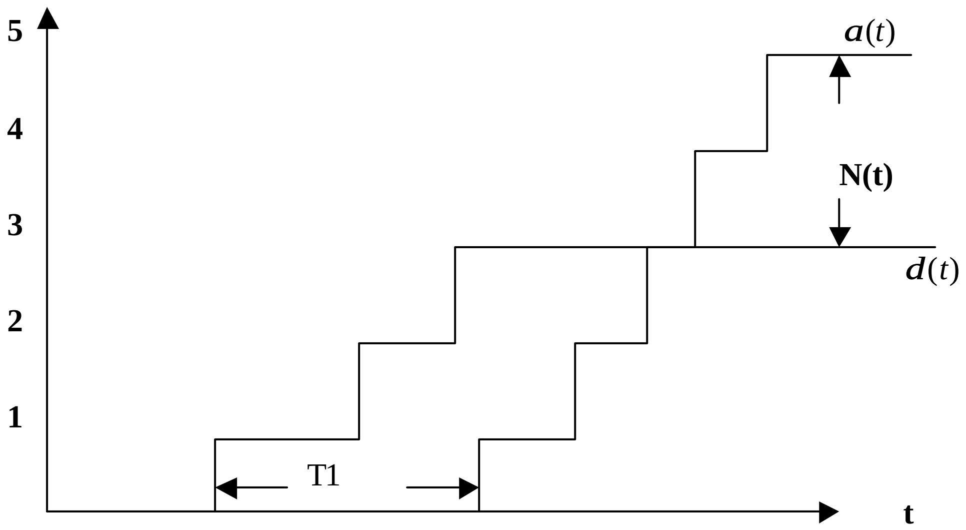
$\overline{N}$:avg # customer in system.

$\overline{T}$:avg time customer spend in system.

λ :avg arrival rate to system.

Thm1: Little's Law

$$\overline{N} = \lambda \cdot \overline{T}$$
$$\overline{Q} = \lambda \cdot \overline{W}$$



$a(t)$ **:# arrival in (0,t)**

$d(t)$ **:#departure in (0,t)**

$$l(t) = \frac{a(t)}{t}$$

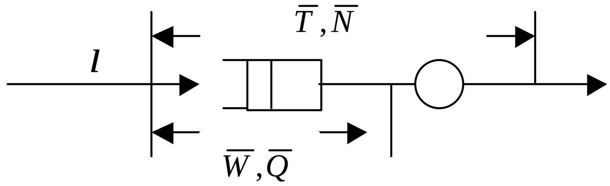
$$N(t) = a(t) - d(t)$$

$r(t)$ **:total time in queue spent by all customer as of T**

$\overline{T}(t)$ **:avg time in queue by customers in (0,t)=** $\frac{r(t)}{a(t)}$

$\overline{N}(t)$ **:avg # customer in system in (0,t)=** $\frac{a(t)\cdot\overline{T}(t)}{t} = l\cdot\overline{T}(t)$

as $t \to \infty$ **,** $\overline{N} = l\overline{T}$



$$\overline{T} = \overline{Q} + \overline{S}$$

$$\overline{N} = \overline{Q} + r$$

$\overline{S}$: *service rate*

r : *utilization of system*

Thm2 Utilization Law

$$r = \frac{avg\ arrival\ rate\ to\ system}{\max\ rate\ at\ which\ system\ can\ handle\ customers}$$

let $\overline{x}$ **be the avg service time if server always busy**

$$r = l\cdot\overline{x}$$

Look at internal (0,t)

$P_0 = prob(server\ is\ idle)$

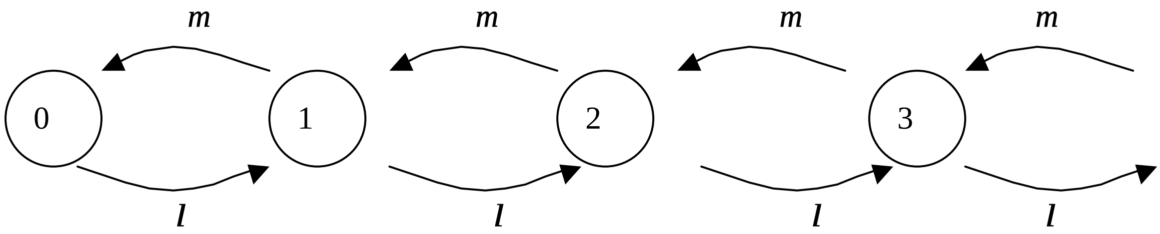
$$t(1-P_0)\frac{1}{x} = l\cdot t$$

$$t\to\infty,\ (1-P_0) = l\cdot\overline{x} \quad\Rightarrow\quad r = 1-P_0$$

M/M/1

- **Poisson arrival w/ rate** l
- **Exp. Dist. Service time w/ rate** m **(avg time** $\frac{1}{m}$**)**
- **Single server**
- **Infinite waiting room**

State diagram



$$P_i = \frac{l}{m} \cdot P_{i-1} = r \cdot P_{i-1} \qquad r = \frac{l}{m}$$

$$\sum_{i=0}^{\infty} P_i = P_0 \cdot \sum_{i=0}^{\infty} r^i = P_0 \cdot \frac{1}{1-r} = 1$$

$$P_0 = 1 - r$$

$$P_i = r^i (1-r)$$

1. **Server utilization** $= 1 - P_0 = r = \frac{l}{m} < 1$

2. $E[N] = \sum_{i=0}^{\infty} i \cdot P_i = \sum_{i=0}^{\infty} i\, r^i (1-r) = (1-r)\frac{r}{(1-r)^2} = \frac{r}{1-r}$

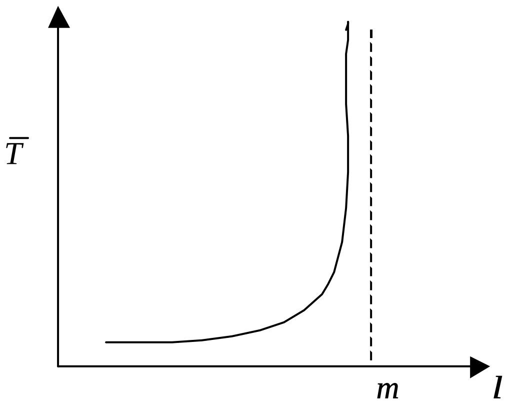
3. $\overline{T} = \frac{\overline{N}}{l} = \frac{\frac{1}{m}}{1-r} = \frac{1}{m-l}$

4. $\overline{W} = \overline{T} - \frac{1}{m} = \frac{1}{m-l} - \frac{1}{m} = \frac{l}{m(m-l)}$

5. $\overline{Q} = l\overline{W} = \frac{l^2}{m(m-l)} = \frac{r^2}{1-r}$

6. $\overline{Q} = \sum_{i=1}^{\infty} (i-1) P_i$

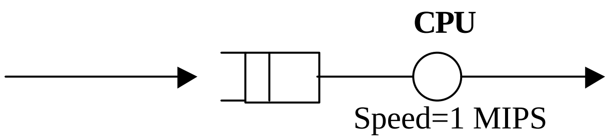
7. $\overline{Q} = \overline{N} - r = \frac{r}{1-r} - r = \frac{r^2}{1-r}$



l 越靠近 m,$\overline{T}$ 上升越快

Example

M/M/!



Each process require exp. Dist.# of instruction with mean 50000

CPU service rate =20= m **jobs/sec**

Avg service time = $\frac{1}{20}$ sec

$$l = 15 \ jobs/\text{sec}$$

$$\overline{T} = \frac{1}{m - l} = \frac{1}{5}\text{sec}$$

$$\overline{N} = l \cdot \overline{T} = 3 \ jobs$$

$$\overline{W} = \frac{1}{m - l} - \frac{1}{m} = \frac{3}{20}\text{sec}$$

$$\overline{Q} = l \cdot \overline{W} = \frac{9}{4} jobs$$

$$r = \frac{15}{20} = \frac{3}{4}$$

Q. What is the avg # customers in queue given there is at least 1

$$E[N \mid N \geq 1]$$

$$= \sum_{i=1}^{\infty} iP[N \mid N \geq 1]$$

$$= \sum_{i=1}^{\infty} i \frac{P[N = i, N \geq 1]}{P[N \geq 1]}$$

$$= \frac{\sum_{i=1}^{\infty} i \cdot P_i}{1 - P_0} = \frac{\overline{N}}{1 - P_0} = \frac{\frac{r}{1 - r}}{r} = \frac{1}{1 - r}$$

Q. What is Prob[more then 10 jobs queued] ?

$$\sum_{i=11}^{\infty} P_i$$