M/M/1 queue

$\pi_i, \overline{N}, \overline{T}, \overline{W}, \overline{Q}$

從 Time average **的觀點來看：**$\overline{T} = \dfrac{1}{\mu - \lambda}$

另一個觀點：Arrival average

<u>PASTA</u>：Poisson Arrival Sees Time Average

$R_k(t)$ **：**probability that an arrival sees k customers at time t

$$R_k(t) = \lim_{\Delta t \to 0} \Pr ob[N(t) = k \mid A(t, t+\Delta t)]$$

$$= \lim_{\Delta t \to 0} \frac{P[N(t) = k, A(t, t+\Delta t)]}{P[A(t, t+\Delta t)]}$$

$$= \frac{P_k(t) \cdot \lambda \cdot \Delta t}{\lambda \cdot \Delta t}$$

$$= P_k(t)$$

$\therefore$ **arrival average = time average (memoryless)**

$$\overline{T} = \sum_{n=0}^{\infty} E[T \mid \text{marked arrival finds n customers in system}]\ \text{P[mark arrival sees n customers in system]} + \frac{1}{\mu}$$

$$= \sum_{n=0}^{\infty} \frac{n}{\mu} \times R(n) + \frac{1}{\mu}$$

$$= \sum_{n=0}^{\infty} \frac{n}{\mu} \times \pi_n + \frac{1}{\mu} \qquad \text{(by PASTA)}$$

$$= \sum_{n=0}^{\infty} \frac{n}{\mu} \times (1-e) \cdot e^{n} + \frac{1}{\mu}$$

$$= \frac{1-\rho}{\mu} \sum_{n=0}^{\infty} n \cdot \rho^{n} + \frac{1}{\mu}$$

$$= \frac{1-\rho}{\mu} \times \frac{\rho}{(1-\rho)^2} + \frac{1}{\mu}$$

$$= \frac{1}{\mu - \lambda}$$

Q:What is the distribution of delay? $(\overline{T} = \frac{1}{m-l})$

A:choose a marked arrival

$$P[T \le t] = \sum_{n=0}^{\infty} P[T \le t \mid \text{n in system upon arrival}]\ \text{P[n in system upon arrival]}$$

(T: n+1 個 exponential 的和)

$$= \sum_{n=0}^{\infty} [\int_0^t m e^{-mx} \frac{(mx)^n}{n!} dx] \cdot (\frac{l}{m})^n \cdot (1 - \frac{l}{m})$$

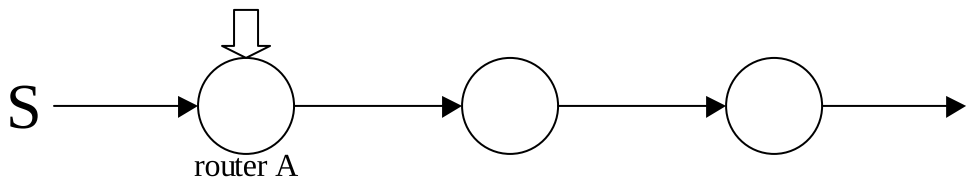
$$\Downarrow \qquad\qquad \Downarrow$$

$$m^{n+1} \cdot \frac{x^n}{n!} \cdot e^{-mx} dx \qquad\qquad \frac{(m-l)\cdot l^n}{m^{n+1}}$$

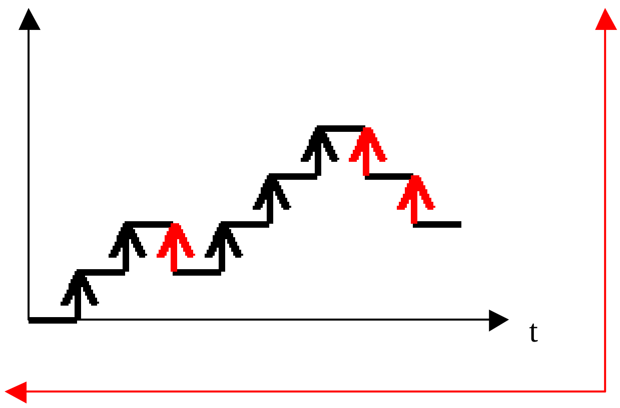
$$= \int_0^t (m-l) \cdot e^{-mx} \sum_{n=0}^{\infty} \frac{(lx)^n}{n!} dx$$

$$= \int_0^t (m-l) \cdot e^{-(m-l)x} dx$$ (Exponential distribution p.d.f.)

$$= 1 - e^{-(m-l)t}$$ (Exponential distribution c.d.f.)



Average departure rate of A：$p_0 - (\frac{1}{l} + \frac{1}{m}) + (1 - p_0) \cdot \frac{1}{m} = \frac{1}{l}$ (Poisson process)



Thm. The departure process of a stationary M/M/1 system is a Poisson process with rate l

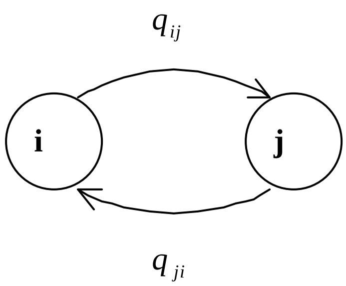
First look : departure $\begin{cases} \text{bury: } \dfrac{1}{m} \\ \text{idle + bury: } \dfrac{1}{l} + \dfrac{1}{m} \end{cases}$

$$\text{average} = (1-p_0)\frac{1}{m} + p_0 \cdot \left(\frac{1}{l} + \frac{1}{m}\right) = \frac{1}{l}$$

Reversibility : A stochastic process, X(t), is reversible if $(X(t_1), X(t_2), \cdots, X(t_m))$ has the same distribution as $(X(t - t_1), X(t - t_2), \cdots, X(t - t_m))$ for any t and $t_1, t_2, \cdots, t_m$

A markov chain is reversible iff "detailed balance equations" are satisfied:

$$p_i q_{if} = p_j q_{ji} \qquad \left(\sum_i p_i q_{ij} = \sum_j p_j q_{ji}\right)$$

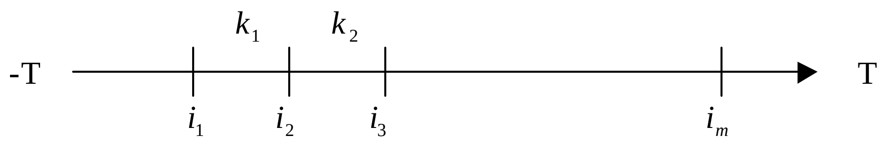


Time reversibility

$$P(X(t_1) = x_1, \cdots, X(t_m) = x_m) = P(X(t - t_1) = x_1, \cdots, X(t - t_m) = x_m)$$

$$p_i q_{if} = p_j q_{ji}$$

$$\sum_i p_i q_{ij} = \sum_j p_j q_{ji}$$



Prob(next state= i_2 | current at i_1, n transmisstion occur)

$$= \frac{q_{i_1 i_2}}{\sum_j q_{ij}} = \frac{q_{i_1 i_2}}{q_{i_1}}$$

When in state i_m, prob that stays for more than k_m unit of time is

$$P(T_m > k_m) = 1 - (1 - e^{-q_{i_m} k_m}) = e^{-q_{i_m} k_m}$$

證明 Time reversibility

$$P(X(t_1) = i_1, \cdots, X(t_m) = i_m)$$

$$= P_{i_1} \cdot [\frac{q_{i_1 i_2}}{q_{i_1}} \cdot q_{i_1} \cdot e^{-q_{i_1} k_1}][\frac{q_{i_2 i_3}}{q_{i_2}} \cdot e^{-q_{i_2} k_2}]\cdots$$

$$= P_{i_1} \cdot q_{i_1 i_2} \cdot q_{i_2 i_3} \cdots q_{i_{m-1} i_m} \cdot e^{-q_{i_1} k_1} \cdot e^{-q_{i_2} k_2} \cdots e^{-q_{i_m} k_m}$$

$$= P_{i_2} \cdot q_{i_2 i_1} \cdot [\cdots]$$

$$= P_{i_m} \cdot q_{i_m i_{m-1}} \cdots q_{i_2 i_1} \cdot [\cdots]$$

$$= P_{i_m} \cdot [\frac{q_{i_m i_{m-1}}}{q_{i_m}} \cdot e^{-q_{i_m} k_m}]\cdots$$

$$= P(X(t - t_1) = i_1, \cdots, X(t - t_m) = i_m)$$

M/M/1/K

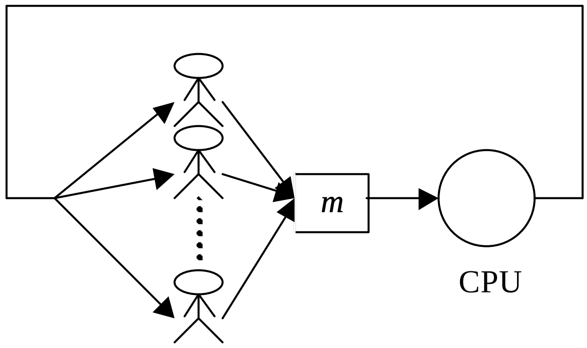
$$p_i = r \cdot p_{i-1}$$

$$p_i = \begin{cases} \dfrac{(1-e)\, r^i}{1 - e^{k+1}} & r < 1 \\ \dfrac{1}{k+1} & r = 1 \end{cases}$$

M/M/1/K/K { machine repairment model / time sharing model

K users: when a user is not waiting for a response, thinks an exponentially distributed time with mean $1/a$, then submit a job.

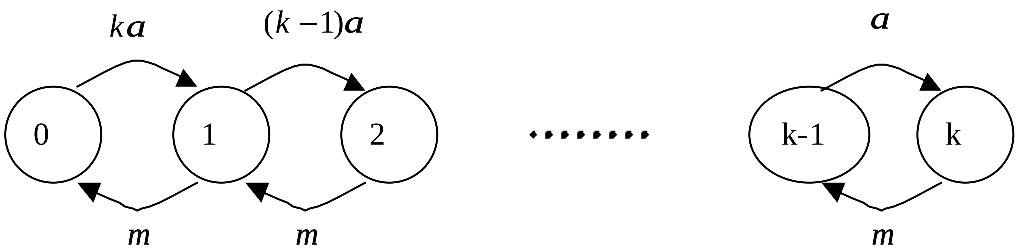
1 CPU: execution time is exponential distribution with mean $1/m$



State descriptor:

Solution 1: $(S_1, S_2, \cdots, S_k)$ $\quad S_i = \{thinking, waiting\}$

Solution 2: (number of jobs at CPU)



$$p_{i+1} = \frac{a(k-i)}{m} \cdot p_i \qquad \sum_{i=0}^{k} p_i = 1$$

$$p_i = \frac{k!}{(k-i)!} \cdot \left(\frac{a}{m}\right)^i \cdot p_0$$

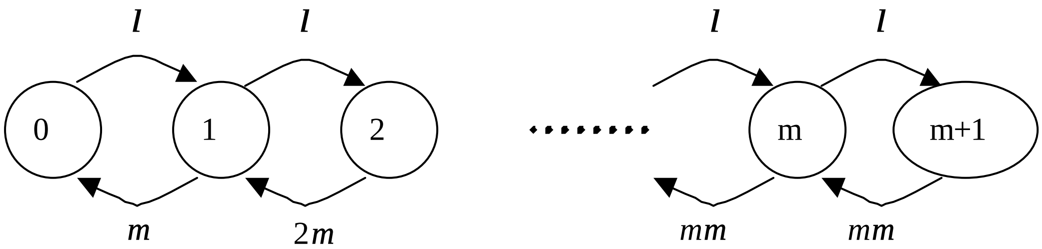
$$p_0 = \frac{\left(\frac{m}{a}\right)^k \big/ k!}{\sum_{i=0}^{k} \left(\frac{m}{a}\right)^i \big/ i!}$$

Server utilization: $1 - p_0$

Throughput: $(1 - p_0) \cdot m$

Queue length: $\sum_{i=0}^{k} i \cdot p_i$

M/M/m



$$e = \frac{l}{m}$$

$$p_1 = r \cdot p_0$$

$$p_2 = \frac{1}{2} \cdot r^2 \cdot p_0$$

$$\vdots$$

$$p_i = \frac{1}{i!} \cdot r^i \cdot p_0$$

$$\vdots$$

$$p_m = \frac{1}{m!} \cdot r^m \cdot p_0$$

$$p_{m+1} = \frac{1}{m!} \cdot \frac{1}{m} \cdot r^{m+1} p_0$$

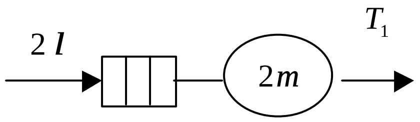
$$\vdots$$

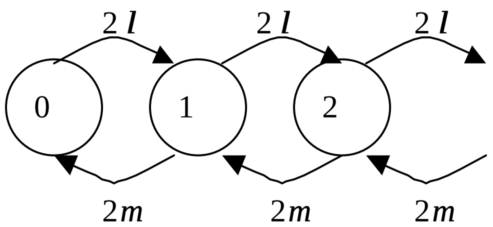
$$p_k = \frac{1}{m!} \cdot \frac{1}{m^{k-m}} \cdot r^k \cdot p_0 \qquad k > m$$

$$p_0 = \frac{1}{\sum_{i=0}^{m} \frac{1}{i!} \cdot r^i + \underbrace{\sum_{i=m+1}^{\infty} \frac{1}{m^{i-m}} \cdot \frac{1}{m!} \cdot r^i}_{\frac{r^m}{m!} \cdot \underbrace{\sum_{j=0}^{\infty} \left(\frac{r}{m}\right)^j}_{\frac{1}{(1-\frac{r}{m})}}}}$$

<u>Erlang C formula (queueing probability)</u> $= \sum_{i=m}^{\infty} p_i = \dfrac{\frac{rm}{m!} \cdot \frac{1}{1-r/m}}{\sum_{i=0}^{m-1} \frac{1}{i!} \cdot r^i + \frac{rm}{m!} \cdot \frac{1}{1-r/m}}$

M/M/1

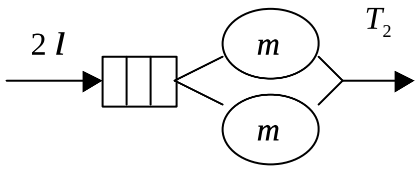


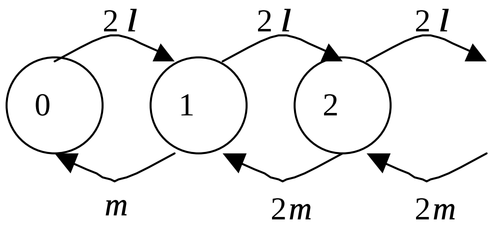


$$\pi_1 = \rho \cdot \pi_0$$
$$\pi_0 = (1-\rho)$$
$$\pi_i = \rho^i (1-\rho)$$

$$\overline{N} = \sum i \cdot \pi_i = (1-\rho)\sum i \cdot \rho^i = \frac{\rho}{1-\rho}$$
$$\overline{T} = \frac{1}{2\lambda} \cdot \frac{\rho}{1-\rho}$$

M/M/2





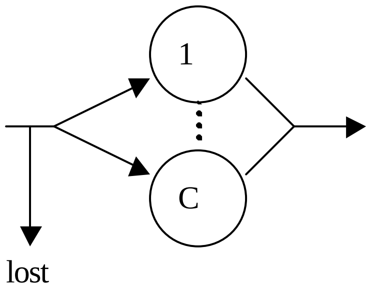
$$\pi_1 = 2\rho \cdot \pi_0, \pi_2 = \rho \cdot \pi_0$$
$$\pi_0 = \frac{1-\rho}{1+\rho}, \pi_1 = 2\rho \cdot \frac{1-\rho}{1+\rho}$$
$$\pi_i = 2\rho^i \cdot \frac{1-\rho}{1+\rho}$$

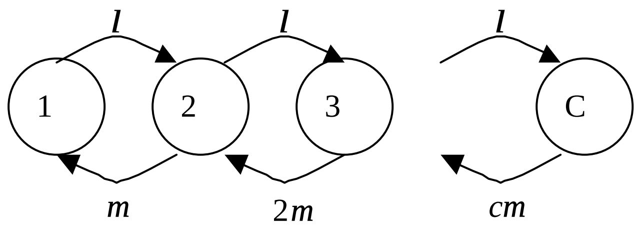
$$\overline{N} = \sum i \cdot \pi_i = \sum i \cdot 2 \cdot \rho^i \cdot \frac{1-\rho}{1+\rho}$$
$$= \frac{2(1-\rho)}{1+\rho} \cdot \sum i \cdot \rho^i = \frac{2\rho}{(1-\rho)(1+\rho)}$$
$$\overline{T} = \frac{1}{2\lambda} \cdot \frac{2\rho}{(1-\rho)(1+\rho)} = \frac{\rho}{\lambda(1-\rho)(1+\rho)}$$

Hint: $\sum i \cdot \rho^{i-1} = \frac{1}{(1-\rho)^2}$

M/M/C/C { **電信網路** / Internet QoS model

Characteristics: 1. admission control
2. blocking
} Loss network





$$p_i = \frac{1}{i!}(\frac{m}{l})^i \cdot p_0 , r = \frac{l}{m}$$

$$p_0 = \frac{1}{\sum_{i=0}^{c} \frac{1}{i!} \cdot r^i}$$

Erlang B formula

Blocking prob. $= p_c = \frac{\frac{1}{c!} \cdot r^c}{\sum_{i=0}^{c} \frac{1}{i!} \cdot r^i}$