# Transform Techniques

Discrete r. v. => Z-transform of pmf

Continuous r.v. => Laplace transform of pdf

## Z-transform：

Let X be a discrete r.v. take value 0,1,2.……,with probability $P_0, P_1, P_2$………

**Def：** Z-transform of X

$$G_x(Z) = \sum_{i=0}^{\infty} P_i Z^i = E[Z^i]$$

**Example；** X is geometrically distribution, $P(x=k) = (1-p)p^k$

$$G_x(Z) = \sum_{i=0}^{\infty} (1-p)p^i \cdot Z^i$$

$$= (1-p)\sum_{1=0}^{\infty} (p \cdot Z)^i$$

$$= \frac{p}{1-pZ} \quad \text{if } pZ<1$$

X is possion distribution $P(x=i) = \dfrac{l^i \cdot e^{-l}}{i!}$

$$G_x(Z) = \sum_{i=0}^{\infty} \frac{l^i \cdot e^{-l}}{i!} Z^i$$

$$= \left[\sum_{i=0}^{\infty} \frac{(l \cdot Z)}{i!}\right] \cdot e^{-l}$$

$$= e^{lZ} \cdot e^{-l} = e^{l(Z-1)}$$

**Moment of X easily complete from Z-transform**

$$G_x(Z) = \sum_{i=0}^{\infty} P_i Z^i$$

$$G_x^{'}(Z) = \sum_{i=0}^{\infty} iP_i \cdot e^{i-1}$$

令 Z=1 帶入得 mean, $G_x^{'}(Z)\Big|_{Z=1} = \sum_{i=0}^{\infty} i \cdot P_i = E[X]$

$$G_X^{''} = \sum_{i=0}^{\infty} i \cdot (i-1) P_i Z^{i-2}$$

$$= \sum_{i=0}^{\infty} i^2 \cdot P_i Z^{i-2} - \sum_{i=0}^{\infty} i \cdot P_i Z^{i-2}$$

令 Z=1 帶入得, $G_x''(Z)\Big|_{Z=1} = \sum_{i=0}^{\infty} i^2 P_i - \sum_{i=0}^{\infty} i \cdot P_i = E\left[X^2\right] - E\left[X\right]$

**<u>Convolution property；</u>**

令 $Y = X_1 + X_2$ 則 $G_{X_1}(Z) = \sum_{i=1}^{\infty} P(x_1 = i) \cdot Z^i$

$$G_{X_2}(Z) = \sum_{i=0}^{\infty} P(x_2 = i) \cdot Z^i$$

$$G_Y(Z) = \sum_{i=0}^{\infty} P(y = x_1 + x_2 = i) \cdot Z^i$$

$$= G_{X_1}(Z) \cdot G_{X_2}(Z)$$

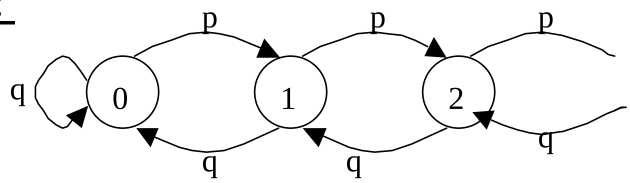
$$G_Y(Z)\Big|_{Z=0} = \sum_{i=0}^{\infty} P_i \cdot Z^i \Big|_{Z=0} = P_0$$

$$G_Y(Z)\Big|_{Z=1} = \sum_{i=0}^{\infty} P_i \cdot Z^i \Big|_{Z=1} = 1$$

**<u>Example：</u>** $e^{l_1(Z-1)} \cdot e^{l_2(Z-1)} = e^{(l_1+l_2)(Z-1)}$

**<u>Random Work ：</u>**

X = i，有 i 元




$$G_X(Z) = \sum_{i=0}^{\infty} p_i \cdot Z^i$$

$$p_1 = \frac{p}{q} \cdot p_0 \quad \Longrightarrow \quad Z^0 \cdot Pp_0 = qp_1 \cdot Z^0$$

$$p_2 = \frac{p}{q} p_1 = (\frac{p}{q})^2 p_0 \qquad Z^1 \cdot Pp_2 = qp_2 \cdot Z^1$$

$$\vdots \qquad\qquad \vdots$$

$$p_i = (\frac{p}{q})^i p_0 \qquad +$$

$$\sum_{i=0}^{\infty} p_i = 1 \;=>\; p_i = (1 - \frac{p}{q})(\frac{p}{q})^i \qquad PG_X(Z) = q\frac{1}{Z}[G_X(Z) - p_0]$$

$$G_X(Z) = \frac{p_0}{1 - \dfrac{pZ}{q}}$$

$$G_X(Z)\Big|_{z=1} = 1 = \frac{p_0}{1 - \dfrac{p}{q}}$$

$$p_0 = 1 - \frac{p}{q}$$

$$p_i = \frac{1}{i!}\frac{d^{(i)}G_X(Z)}{dz}\Bigg|_{z=0}$$

$$G_X(Z) = \frac{1 - \dfrac{p}{q}}{1 - \dfrac{pZ}{q}} \quad , \quad E[X] = G_X^{'}(Z)\Big|_{Z=0} = \frac{p}{q-p}$$

## Laplace Transform

Let X be a continuous r.v. with pdf $f_X(x)$

**Def**：Laplace transform of X

$$F_X^*(S) = \int_0^\infty f_X(x)e^{-Sx}dx$$

**Example**：

$$F_X^*(S) = \int_0^\infty l \cdot e^{-lx}e^{-sx}dx$$

$$= \int_0^\infty l \cdot e^{-(l+S)x}dx$$

$$= \frac{l}{l+S}$$

**Moment**：

$$E[X] = \frac{-d}{dS}F_X^*(S)\Big|_{S=0}$$

$$E[X^i] = (-1)^i \frac{d}{dS}F_X^*(S)\Big|_{S=0}$$

**Convolution**：

if $X_1, X_2, ........, X_n$ are independent continuous r.v.

with transform $F_{X_1}^*(S), F_{X_2}^*(S), ..........., F_{X_n}^*(S)$

$$Y = X_1 + X_2 + \ldots\ldots\ldots\ldots + X_n$$

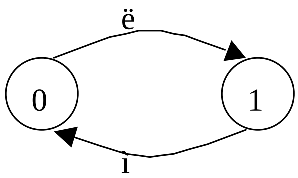
$$F_Y^*(S) = F_{X_1}^*(S) \cdot F_{X_2}^*(S) \cdot \ldots\ldots\ldots \cdot F_{X_n}^*(S)$$

**<u>Example：</u>** exponential distribution

$$F_X^*(S) = \left(\frac{l}{l+S}\right)^n \quad => \quad f_Y(y) = \frac{l(lx)^{n-1}}{(n-1)!} e^{-lx} \ldots\ldots \text{gamma distribution}$$

if n **為正整數則為** Erlang distribution

**<u>Diff. Equation</u>：** solved by Laplace Transform




$$P_{00}'(t) = \sum_{k=0}^{\infty} P_k(t) q_{k0}$$

$$= P_{00}(t) \cdot (-l) + P_{01}(t) \cdot m$$

$$P_{00}'(t) = -(l+m) P_{00}(t) + m$$

$$SP_{\infty}^*(S) - P_{00}(0) = -(l+m) P_{00}^*(S) + \frac{m}{S}$$

$$P_{\infty}^*(S) = \frac{\frac{l+m}{S}}{S + l + m}$$

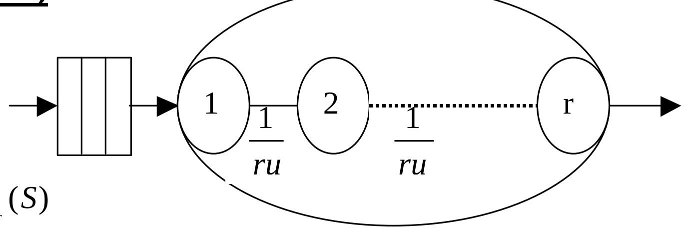
$$= \frac{S+m}{S(S+l+m)}$$

$$= \frac{1}{S} \cdot \frac{m}{l+m} + \frac{1}{S+l+m} \cdot \frac{l}{l+m}$$

$$\therefore P_{00}(t) = \frac{m}{l+m} + \frac{l}{l+m} e^{-(l+m)t}$$

## <u>M/Er/1 (Erlang distribution)</u>

Er/M/1



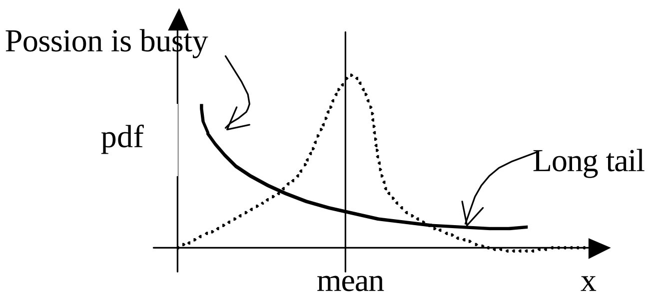

$$Y = X_1 + X_2 + \ldots + X_r$$

$$F_Y(S) = F_{X_1}(S) \cdot F_{X_2}(S) \cdot \ldots\ldots \cdot F_{X_r}(S)$$

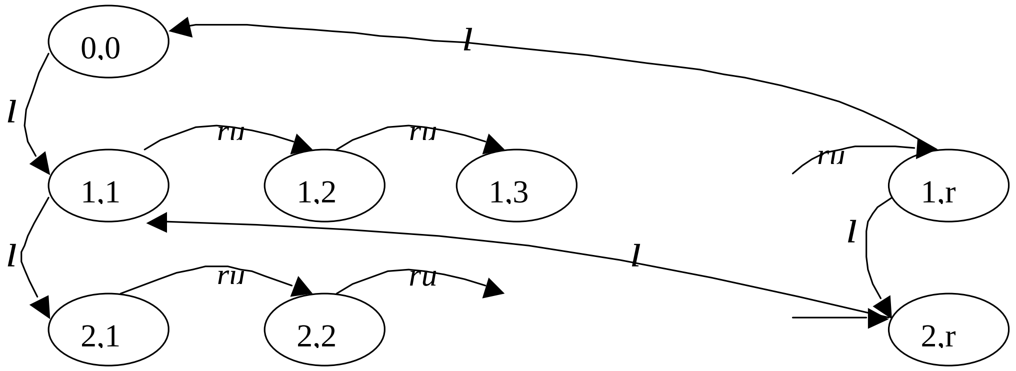
$$E[X] = r \cdot \frac{1}{r+u} = \frac{1}{u}$$

$$s_X^2 = r \cdot \left(\frac{1}{ru}\right)^2 = \frac{1}{ru^2}$$

$$b(x) = \frac{ru(ru \cdot x)^{r-1} \cdot e^{-rux}}{(r-1)!}$$

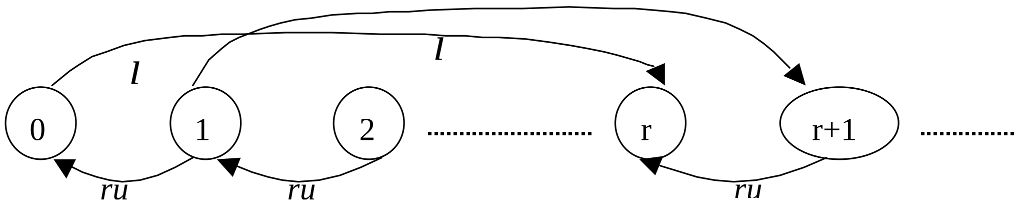


(# queue, stage in service)




(n-1)r+(r-I+1)

# queue, stage in service => # stage table served




$P_0 = p_0$ $\quad$ $P_1 = p_1 + \ldots + p_r$ $\quad$ $P_1 = p_{r+1} + .. + p_{2r}$

$$l \cdot p_0 = ru \cdot p_1$$

$$(l + ru)p_j = l \cdot p_{j-r} + ru \cdot p_{j+1}$$

$j \geq r, Let p_{i=0}$ for i<0

$$(l + u)p_j \cdot Z^j = l p_{j-r} \cdot Z^j + ru \cdot p_{j+1} - Z^j$$

$$(l + ru)\sum_{j=1}^{\infty} p_j Z^j = lZ^r \sum_{j=0}^{\infty} p_j Z^j + ru\frac{1}{Z}\sum_{j=1}^{\infty} p_{j+1} Z^{j+1}$$

$$(l + ru)[P(Z) - p_0] = lZ^r P(Z) + ru\frac{1}{Z}[P(Z) - p_0 - p_1 Z]$$

$$P(Z) = \frac{p_0(l + ru - \frac{ru}{Z}) - rup}{l + ru - lZ^r - \frac{ru}{Z}}$$

boundary condition： $lp_0 = rup_1$

$$P(Z) = \frac{p_0 - ru(1 - Z)}{ru - lZ^{r+1} - (l + ru)Z}$$

$$P(Z)\big|_{Z=1} = 1 = \frac{0}{0}$$

L' Hopital' s rule

$$\lim_{x->c}\frac{f(x)}{g(x)} = \lim_{x->c}\frac{f'(x)}{g'(x)}$$

if $\frac{f(x)}{g(x)} = \frac{0}{0}, or, \frac{\infty}{\infty}$ and

$g'(x) \neq 0$

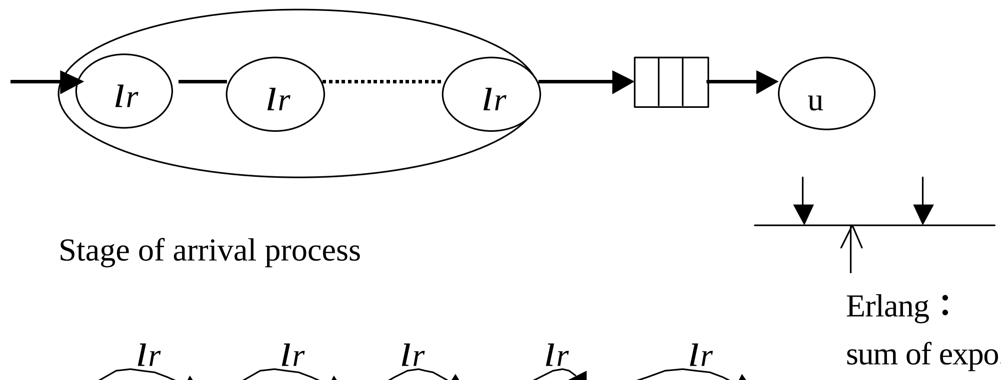
$P(Z)|_{Z=1} = 1$ by L' Hoptial' s rule

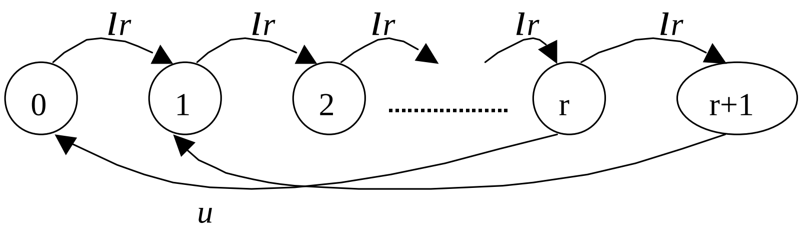
$$P^{'}(Z)|_{Z=1} = \frac{-p_0 + u}{(r+1)\,l - (\,l + ru)}\ \Big|_{Z=1}$$

$$= \frac{ru + p_0}{r(u - l)}$$

$$= \frac{u}{u - l} p_0 \qquad \therefore p_0 = \frac{u - l}{u} = 1 - \frac{l}{u},\ l < u$$

## Er/M/1 Queue



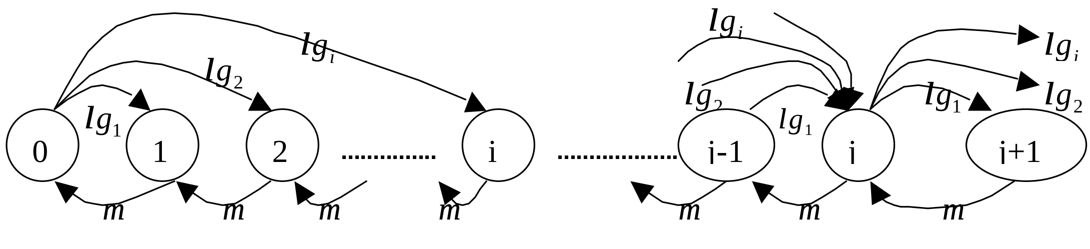

Bulk arrival system — all customers have exponential service times.

— customers arrival groups.

— interarrival time between arrival of groups is exponential distribution with mean $\frac{1}{l}$

— Let $g_i$ ：probability that there are i in an arriving group

# of tasks in system

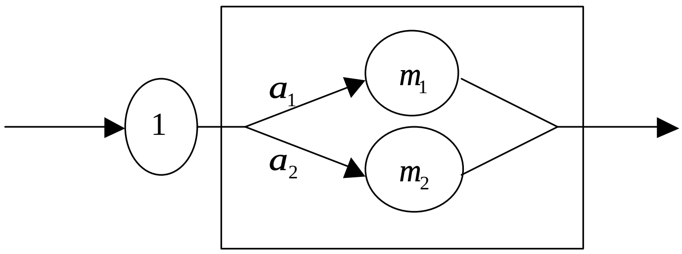


$$p_j(\sum_{i=1}^{\infty} lg_i + m) = mp_{j+1} + \sum_{i=0}^{j-1} p_j\, l \cdot g_{j-1}$$

$$p_j(l + m) = m \cdot p_{j+1} + \sum_{i=0}^{j-1} p_i\, l \cdot g_{j-i}$$

Model service time that are more variable than exponential.

Hyper exponential distribution




$$b(x) = a_1 \cdot m_1 e^{m_1 x} + a_2 \cdot m_2 e^{m_2 x}$$

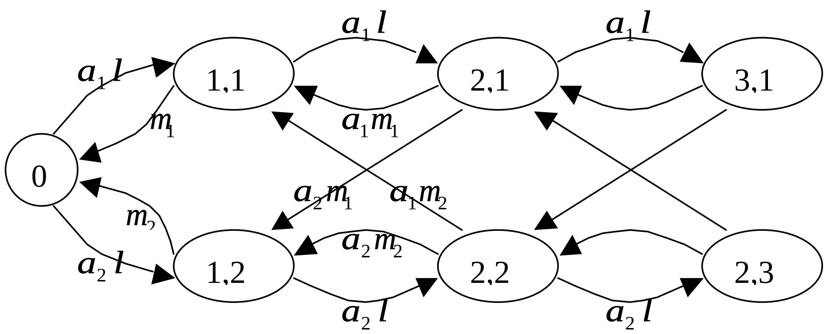
$$B^x(S) = \frac{a_1 m_1}{S + m_1} + \frac{a_2 m_2}{S + m_2}$$

$$E[X] = \frac{a_1}{m_1} + \frac{a_2}{m_2}$$

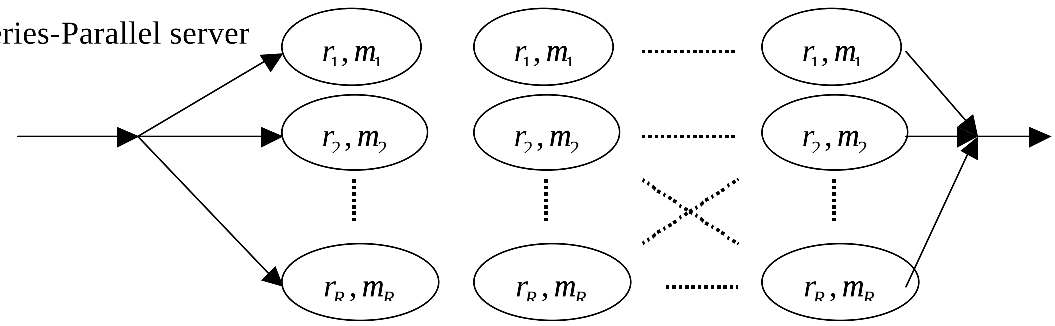
$$E[X] = 2 \cdot [\frac{a_1}{m_1} + \frac{a_2}{m_2}] \qquad s^2 = E[X^2] - E[X]^2$$

## M/H2/1 Queue

(# jobs, which server in service)




Series-Parallel server

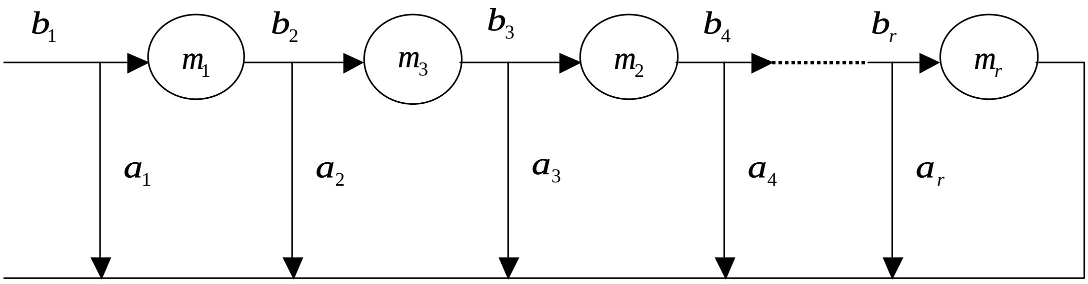


$$b(x) = \sum_{i=0}^{R} a_i \cdot \frac{r_i m_i (r_i m_i)^{r-1}}{(r_i - 1)!} \cdot e^{-r_i m_i x}$$

$$B^*(S) = \sum_{i-1}^{R} a_i \cdot (\frac{r_i m_i}{S + r_i m_i})^{r_i}$$

$$B^*(S) = \sum_{i-1}^{R} a_i \cdot \prod_{j-1}^{r_i} \frac{m_{ij}}{S + m_{ij}}$$

## <u>Coxian distribution</u>




$$B^*(S) = a_1 + \sum_{i=1}^{r} b_1 ...... b_i a_{i+1} \prod_{j=1}^{i} (\frac{m_j}{S + m_j})$$